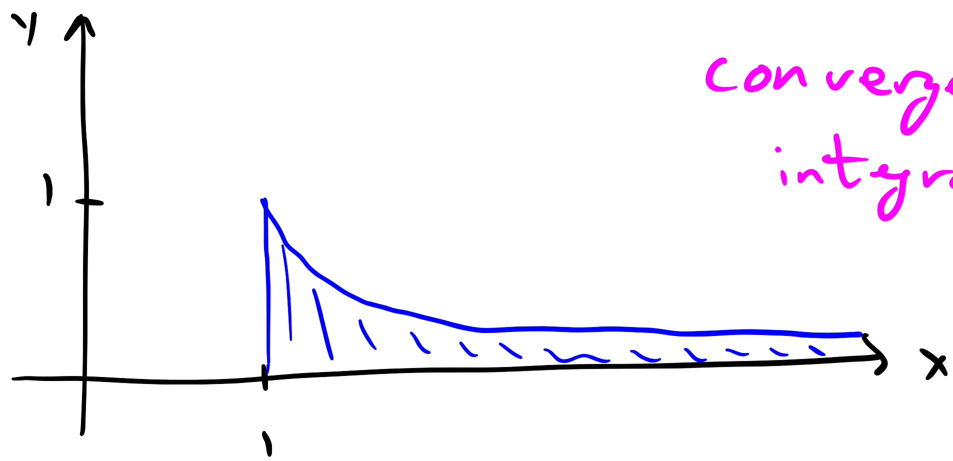
# Improper Integrals

Ex: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx$

$$= \lim_{t\to\infty}\left(\frac{-1}{x}\Big|_1^t\right) = \lim_{t\to\infty}\left(\frac{-1}{t} - \left(\frac{-1}{1}\right)\right) = 1$$




convergent integral

# Type 1 Improper Integrals

ⓐ $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$

provided the limit exists

ⓑ $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$

provided the limit exists

If these integrals exist (limit is finite), we say the integral coverges. If an integral doesn't converge, we say it diverges.

ⓒ $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

We can use any real number c.

Ex: ① $$\int_{1}^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_{1}^{t} \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \left( \ln x \Big|_{1}^{t} \right) = \lim_{t\to\infty} \left( \ln t - \cancel{\ln 1}^{\,0} \right) = \infty$$

$$\int_{1}^{\infty} \frac{1}{x}\,dx = \infty$$ divergent integral

② $$\int_{0}^{\infty} e^{-2x}\,dx = \lim_{t\to\infty} \int_{0}^{t} e^{-2x}\,dx = \lim_{t\to\infty} \left( -\frac{1}{2} e^{-2x} \Big|_{0}^{t} \right)$$

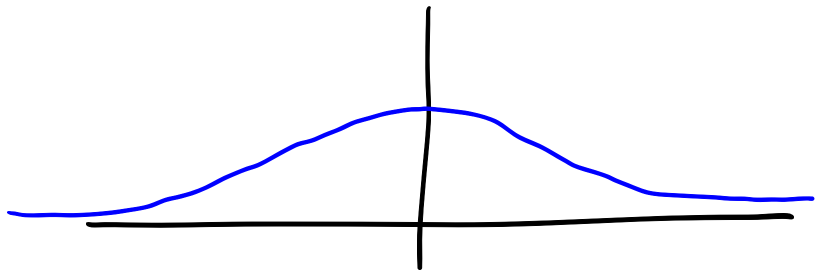
$$= \lim_{t\to\infty} \left( -\frac{1}{2} e^{-2t} - \left( -\frac{1}{2} e^{0} \right) \right) = 0 + \frac{1}{2} = \boxed{\frac{1}{2}}$$

(3) $$\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x}}\,dx = \lim_{t\to-\infty} \int_{t}^{-1} x^{-1/3}\,dx$$

$$= \lim_{t\to-\infty} \left( \frac{3}{2} x^{2/3} \Big|_{t}^{-1} \right) = \lim_{t\to-\infty} \left( \frac{3}{2} - \frac{3}{2} t^{2/3} \right)$$

$$= \frac{3}{2} - \infty = \boxed{-\infty}$$

(4) $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$



$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \left( \arctan x \Big|_{t}^{0} \right) + \lim_{t\to\infty} \left( \arctan x \Big|_{0}^{t} \right)$$

$$= \lim_{t\to-\infty} \left( \cancelto{0}{\arctan 0} - \arctan t \right) + \lim_{t\to\infty} \left( \arctan t - \cancelto{0}{\arctan 0} \right)$$

$$= -\left( \frac{-\pi}{2} \right) + \frac{\pi}{2} = \pi$$

Ex: $$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \lim_{t\to 2^+} \left( 2\sqrt{x-2} \,\Big|_t^5 \right) = \lim_{t\to 2^+} \left( 2\sqrt{3} - 2\sqrt{t-2} \right)$$

$$= 2\sqrt{3} - 2\sqrt{0} = \boxed{2\sqrt{3}}$$

## Type 2 Improper Integrals

ⓐ If $f$ is continuous on $[a,b)$ and discontinuous at $b$, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

provided the limit exists.

ⓑ If $f$ is continuous on $(a,b]$ and discontinuous at $a$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

provided the limit exists.

⊙ If $f$ has a discontinuity at $x=c$, $a<c<b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

provided the right hand side integrals exist.